J. Fluid Mech. (1980), *vol.* 81, *part* 1, *pp.* 171–191
Printed in Great Britain

The response of sheared turbulence to additional distortion

By A. A. TOWNSEND
Emmanuel College, Cambridge

(Received 9 March 1979 and in revised form 23 August 1979)

In unidirectional flows, the ratios of Reynolds shear stress to total intensity (except near positions of zero stress) remain remarkably constant from one flow to another, but curvature or strong divergence of the mean flow causes very considerable changes in the stress ratios. A scheme for calculating the changes is described, based on the rapid-distortion approximation of the equations of motion. The results depend to some extent on the effective history of distortion of the turbulence and on the magnitude of an eddy viscosity that models the effect of nonlinear transfer of energy to smaller eddies of the dissipation sequence, but the correspondence with measured values in a distorted wake and in a curved mixing layer is fairly good. In particular, the curious behaviour of stress ratios in the curved mixing-layer can be reproduced qualitatively without any difficulty. Small perturbations of wall turbulence provide a simple application, and earlier calculations of the energy transfer between wind and water waves have been repeated including the changes in the stress ratios predicted by the scheme. In the latter case, very large changes in the distributions of pressure and shear stress are found, and the rates of energy transfer are much larger and in better agreement with observations.

1. Introduction

Over the past decade, predictive theories of turbulent shear flow have been constructed from the equations for the Reynolds stress tensor by making semi-empirical assumptions about the distributions of viscous energy dissipation and flux of turbulent energy and, in particular, about the relative magnitudes of components of the stress tensor. If the distortion by the mean flow is almost unidirectional simple shearing and so satisfies the conditions for the 'boundary layer' approximation, good agreement between prediction and measurement has been obtained for important properties of boundary layers such as mean velocities, wall stress and heat transfer. Prediction of turbulent intensities is less satisfactory, but the normal Reynolds stresses have little influence on these flows and the good agreement with the predictions is ensured by using assumptions consistent with the 'law of the wall'. For free turbulent flows, the predictions are more sensitive to details of the basic assumptions but the variation of entrainment rates between wakes, jets and mixing-layers is well described by the scheme (Townsend 1976).

If the mean flow is appreciably three-dimensional or strongly divergent, straightforward extensions of the theories may be seriously in error, and several attempts have been made to improve the predictive ability by introducing additional parameters to describe the effects of the more complex flow distortions on the turbulent motion. One example is the use by Bradshaw (1969) of an analogy between the effects of buoyancy in unidirectional flow and the effects of angular momentum transfer in flow with curved streamlines. Another approach is that of Launder, Reece & Rodi (1975) who use the equations for the Reynolds stresses to calculate the stress ratios, essentially by regarding the pressure–velocity terms as acting to remove anisotropy induced by the inertial terms. While the equations describe many flows with some accuracy, experiments on the distortion of homogeneous turbulence agree with theoretical calculations in showing that the functions of the inertial and pressure terms cannot be clearly distinguished. Tucker & Reynolds (1968) have measured intensity ratios during irrotational plane straining of grid turbulence, and their results show that redistribution of energy among the velocity components occurs simultaneously with the distortion and that the return to isotropy is rather slow after removal of the mean flow distortion.

While the Reynolds stress equations are exact, they describe only some aspects of the turbulent velocity field and can say very little about the flow patterns, in particular about the form and orientations of the energy-containing eddies. If the details of the motion turn out to be relevant for some unusual forms of flow distortion, a calculation scheme that does not include the necessary information will not give correct results. One such flow is the curved mixing-layer studied by Castro & Bradshaw (1976). As the mixing-layer begins to change direction, the first effect of the curvature is an expected decrease in the stress-intensity ratio but, after the flow direction has turned through 45°, the ratio increases and recovers to nearly its initial value after a change of direction by 90°.

In many kinds of unidirectional flow, the measured velocity correlations are, except for contributions from eddies characteristic of the particular flow, remarkably similar, and the ratios of Reynolds shear stress to total turbulent intensity are also much the same. In addition, the velocity correlations resemble closely those calculated from the rapid-distortion equations for initially isotropic turbulence that has undergone total shears between two and three (Townsend 1970). For the purposes of this paper, it is not necessary to discuss the reasons for the ability of the rapid-distortion calculation to provide an acceptable description of the correlation function in unidirectional shear flow. The calculation simply and conveniently provides an adequate specification of the motion. If a turbulent flow passes suddenly from a region of unidirectional shear to one of more complex distortion, the initial response will be described by the rapid-distortion equations and the changes in the stress ratios could be calculated. Here, the possibility will be examined, that the rapid-distortion equations can be used to calculate stress ratios for finite general distortions of sheared turbulence.

The rapid-distortion approximation has been used for a long time, particularly to calculate the effect of the irrotational distortion on turbulence passing through a contraction or around a bluff body (Batchelor & Proudman 1954; Hunt 1973). Then the changes depend only on the total distortion by the mean flow. Within a developing shear flow, the eddies exist within mean velocity gradients that are strongly rotational and the changes are influenced by the history of velocity gradient as well as the total distortion and rotation unless the type of gradient does not change. Results for continuing simple shearing are given by Moffatt (1965) and by Deissler (1965).

After a description of the rapid-distortion calculation procedure, stress ratios calculated for the curved mixing-layer and for a three-dimensionally distorted wake are compared with measured values, and a calculation of the flow over water waves in a deep turbulent boundary layer is made. For the latter purpose, the equation for the turbulent kinetic energy is used to close the system of equations. In its present form, the rapid-distortion scheme is limited to prediction of stress ratios, and some such auxiliary equation is needed before it could form part of a complete procedure for the calculation of turbulent flows.

2. The effects of mean flow distortion on the turbulent motion

Although it is usual to discuss the structure of turbulent shear flows as though the eddies were affected only by local velocity gradients and by eddies in their immediate neighbourhood, observed velocity patterns always show strong coherence over the full width of the flow. Some account of the large extent of the patterns may be taken (i) by postulating eddy diffusion (or convection) rates for energy and Reynolds stress, and (ii) by regarding each eddy as a local structure that moves in the flow, either systematically with the mean flow velocity at its current position, erratically in the velocity fields of neighbouring eddies, or with a velocity induced by its own vorticity.

On this view, each eddy has been subjected to a total strain that is the rate of strain experienced by the eddy integrated over its past history. Since eddies significantly smaller than those contributing most to the kinetic energy and Reynolds stress experience random turbulent rates of strain as well as strain by the mean flow, their total strains are much more variable and in sum they form part of a 'background' of roughly isotropic turbulence. Only the energy-containing 'large' eddies are strained almost entirely by the mean flow gradients, and it is their history that matters for the development of Reynolds stresses. In simple free turbulent flows, eddies originate as the fluid composing them is entrained from the ambient flow, but the trajectories of eddies at a particular point in the flow are not all the same and the total strains are different. The effective total strain is an average, and the effect may be modelled by defining an eddy diffusion coefficient for strain (Townsend 1970). In wall turbulence, the effective duration of continuous straining is limited by the 'bursting' phenomenon, which means that eddies comparable in diameter with distance from the surface remain in environments of strong shearing for limited periods of time. In both kinds of flow, the distortion is very nearly a simple shearing and the effective distortion is finite. In curved or divergent flows, the distortion is more complex and the history of distortion over the life-time of the eddy must be used rather than a current effective distortion.

The rapid-distortion equations to be used refer to homogeneous turbulent flow in a uniform gradient of mean velocity, implying that the length scale for the turbulence is small compared with the flow width, and they assume that transfer of energy by turbulence–turbulence interactions has little effect on the flow patterns of the large, energy-containing eddies. The interactions *are* responsible for the transfer of energy to the smaller eddies which contain only a small part of the total turbulent energy and which are less anisotropic than the large eddies. Physically, the implication is that large eddies are stable flow patterns which do not overlap one another and which change almost entirely as a result of interaction with the mean flow gradient. Evidence of the discrete nature and stability of the large eddies is found in the slow approach to isotropy and the long persistence of flow patterns in flows without gradients of mean velocity (Townsend 1976, chap. 3).

If the effects of the turbulence–turbulence interactions are nearly limited to transfer of energy to the smaller eddies of the turbulence, the transfer will be roughly uniform over the extent of a large eddy and a simple way to describe it is by an eddy viscosity acting on the large eddies. For any particular sequence of distortion by the mean flow, it is found that the calculated stress-ratios change very little while the magnitude of the eddy viscosity changes considerably, and it is improbable that a different modelling of the transfer would give different results. Obviously, the magnitude of the viscosity coefficient has a direct and considerable effect on the calculated intensities, but the rapid-distortion scheme is intended to predict changes in flow patterns and would need considerable development before intensities could be specified.

If the velocity field within a finite but large volume is given by the three-dimensional Fourier series,

$$u_i(\mathbf{x}, t) = \Sigma\, a_i(\mathbf{k}, t) \exp(i\mathbf{k}\,.\,\mathbf{x}), \tag{2.1}$$

the rapid-distortion equations of motion for a uniform gradient of mean velocity, $\partial U_i/\partial x_j$, are

$$\frac{\partial a_i}{\partial t} + \nu_T\, k^2 a_i = \left(2\frac{k_i k_j}{k^2}\frac{\partial U_j}{\partial x_r} - \frac{\partial U_i}{\partial x_r}\right) a_r, \tag{2.2}$$

with

$$\frac{dk_i}{dt} = -k_j \frac{\partial U_j}{\partial x_i} \tag{2.3}$$

to describe the distortion and rotation of the wavenumber lattice by the mean flow (Pearson 1959). The omitted nonlinear terms have been replaced by the 'viscous' term, $\nu_T\, k^2 a_i$. The equations have an integrating factor,

$$V(\mathbf{k}, t) = \exp \int \nu_T\, k^2\, dt \tag{2.4}$$

and solutions of equation (2.2) can be obtained by dividing solutions of the equations without the viscous term by $V(\mathbf{k}, t)$.

To find values of the Reynolds stresses, it is necessary to specify the strain history of the turbulent fluid from some virtual origin of time when it was nearly isotropic and to assign a value to the eddy viscosity. Consider first the changes of wavenumber for a single component. Initial and final values, $\mathbf{k}'$ and $\mathbf{k}$, are linearly related by

$$k_i = K_{ij}\, k'_j, \tag{2.5}$$

where

$$dK_{ij}/dt = -K_{pj}\, \partial U_p/\partial x_i \tag{2.6}$$

with initial values, $K'_{ij} = \delta_{ij}$. Similarly, initial and final values of the Fourier coefficients are linearly related by

$$a_i = A_{ij}\, a'_j, \tag{2.7}$$

where

$$\frac{dA_{ij}}{dt} = \left(2\frac{k_i k_r}{k^2}\frac{\partial U_r}{\partial x_q} - \frac{\partial U_i}{\partial x_q}\right) A_{qj} \tag{2.8}$$

with the initial condition, $A_{ij} = \delta_{ij}$.

Values of the Reynolds stresses are found by forming the spectrum function,

$$F_{ij}(\mathbf{k}) = N\langle a_i(\mathbf{k})\, a_j^*(\mathbf{k}) + a_i^*(\mathbf{k})\, a_j(\mathbf{k})\rangle,$$

where N is a normalizing factor, and then

$$F_{ij}(\mathbf{k}) = A_{ip}\, A_{jq}\, F'_{pq}(\mathbf{k}'). \tag{2.9}$$

The initial turbulence is assumed to be isotropic, so that

$$F'_{pq}(\mathbf{k}') = (\delta_{pq} - k'_p\, k'_q/k'^2)\, \phi(k')$$

and
$$\langle u_i\, u_j\rangle = \int_0^\pi \int_0^{2\pi} A_{ip}\, A_{jq}(\delta_{pq} - k'_p\, k'_q/k'^2) \sin\theta\, d\theta\, d\phi \int_0^\infty \phi(k')\, k'^2\, dk', \tag{2.10}$$

where
$$\int_0^\infty \phi(k')\, k'^2\, dk' = \frac{3}{4\pi}\langle u'^2\rangle. \tag{2.11}$$

Without the eddy transfer term the results are independent of the spectral form, but a form must be assumed if the transfer is to be included. Since the energy-containing eddies are considered to be fairly homogeneous in size, the error law spectrum defined by

$$\phi(k') = Ck'^2 \exp(-\tfrac{1}{2}k'^2 L^2)$$

gives an appropriate representation of the initially isotropic turbulence. In the polar co-ordinates, the exponent of the integrating factor may be written as

$$\frac{\nu_T\, t_d}{L^2}\, k^2 L^2 Q(\theta, \phi), \tag{2.12}$$

where
$$Q(\theta, \phi) = \int_0^{t_d} (k^2/k'^2)\, dt/t_d$$

is an average value of (k^2/k'^2) over the duration of the distortion, t_d.

Making allowance for transfer of energy from the main eddies, the expression for the Reynolds stresses becomes

$$\langle u_i\, u_j\rangle = \int_0^\pi \int_0^{2\pi} A_{ir}\, A_{jq} \left(\delta_{rq} - \frac{k'_r\, k'_q}{k'^2}\right) \left(1 + 4\frac{\nu_T\, t_d}{L^2} Q\right)^{-\frac{5}{2}} \sin\theta\, d\theta\, d\phi\, \frac{3}{8\pi}\langle u_1'^2\rangle \tag{2.13}$$

and requires an estimate of the non-dimensional eddy transfer coefficient, $\nu_T\, t_d/L^2$.

Experience with calculations for flows with unidirectional shearing shows that the calculated *ratios* of the stress components are nearly independent of the value of the transfer coefficient, provided that it is not unrealistically small. Its magnitude may be estimated by observing that production and dissipation of turbulent energy are roughly equal in typical shear flows. Although the large eddies of a shear flow are far from isotropic, the initial transfer of energy from eddies with the error-law spectrum caused by an eddy transfer coefficient ν_T is

$$\epsilon = 15\nu_T \langle u_1^2\rangle / L^2$$

and is not expected to change greatly during distortion.

For simple shearing at a rate $\dot{\alpha}$, energy is transferred from the mean flow at a rate $\tau\dot{\alpha}$, where τ is the Reynolds shear stress. In unidirectional shear flows, the ratio of shear stress to total turbulent intensity is near to 0·15 and so

$$\frac{\nu_T}{\dot{\alpha} L^2} \approx 0{\cdot}03. \tag{2.14}$$

If the effective total shear is $\dot{\alpha}t_d = \alpha_0$,

$$\nu_T t_d/L^2 \doteqdot 0{\cdot}03\,\alpha_0. \tag{2.15}$$

Both for simple and complex distortions, suitable values of the non-dimensional transfer coefficient should be found in the range 0·1–0·2.

Explicit solutions of the equations for the transformation matrices are easily found for simple types of distortion, in particular, for plane shearing, for irrotational distortion with the principal axes fixed in space, for irrotational, constant-circulation, curved flow, and for solid-body rotation.† It would be convenient if the transformation matrices for a complex history of distortion were the product of matrices for any sequence of simple distortion that together produce the total distortion. Unfortunately, this commutative property holds only for the wavenumber transformation and is not valid for the coefficient transformation if the distortion includes any rotation.

For any mean flow containing vorticity, the transformation matrix for the coefficients should be obtained by integration of equation (2.8) with the appropriate mean velocity gradients. To simplify the numerical work, the integrations for the distorted wake and for the curved mixing-layer have been done by 'quadrature', that is, the distortion history has been split into a number of steps, each of simple type and not too large, and chosen so that the total distortion during as well as at the end of each step is reasonably close to the actual flow distortion. The adequacy of the results can be tested by reducing the step size. The final Fourier coefficients are then calculated from the original ones with a transformation matrix that is the product of the matrices for the component steps of the total distortion, e.g. for a distortion with transformations A, B, C in that order,

$$a_i(\mathbf{k}) = C_{ij}\,B_{jk}\,A_{kp}\,a'_p(\mathbf{k}'). \tag{2.16}$$

Transformations for four simple forms of distortion are given in the appendix.

3. The distorted wake

Measurements of turbulent intensities and stresses have been made in the wake of a circular cylinder as it passes through a distorting duct. The overall effect of the duct is to stretch the wake in the direction of shear, i.e. in the direction normal to the plane of the wake, and to compress it in the plane of the wake without significant longitudinal extension. However, considerable longitudinal extension and compression occurs during the passage through the distorting section (figure 1). For the calculation of stress ratios, the lateral extension ratios were assumed to be proportional to the local dimensions of the duct section, and the longitudinal extension ratios found from the distribution of velocity on the centre-line. The cylinder diameter is 3·19 mm and the flow velocities near 9 m s^{-1}, with cylinder Reynolds numbers near 2000. Intensities and Reynolds stresses were measured over traverses at distances between 41 and 116 cm from the cylinder, beginning upstream of the distorting section and ending downstream of it. Details of the experimental arrangements are given by Elliott (1976) and a fuller account is being prepared for publication.

† For convenience, solid-body rotation is included in the term 'distortion' since it induces considerable changes in the turbulent motion.

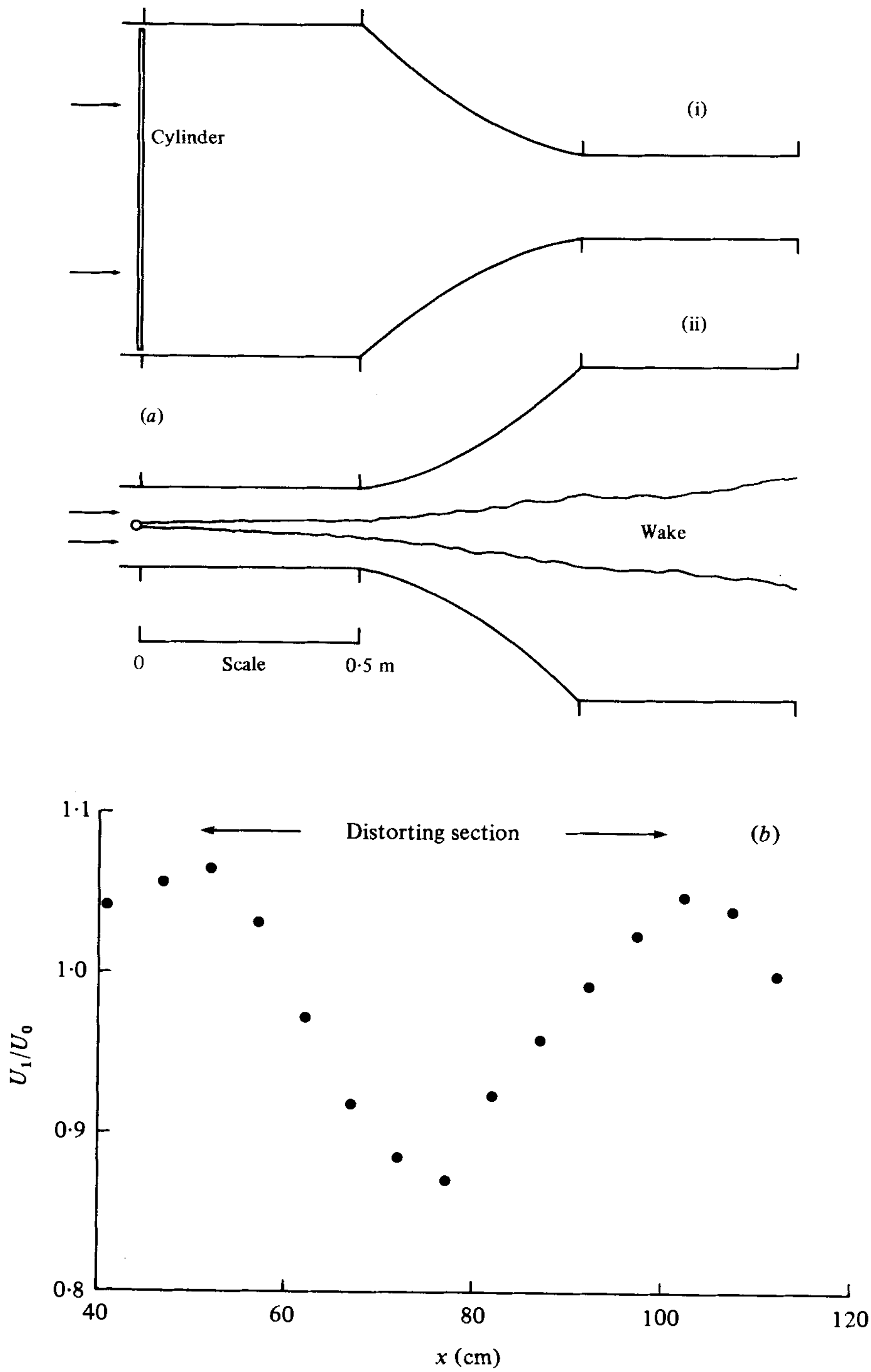


FIGURE 1. The distorted wake flow. (*a*) Longitudinal sections of the distorting duct: (i) vertical; (ii) horizontal. (*b*) Distribution of mean velocity along the centre-line of the duct.

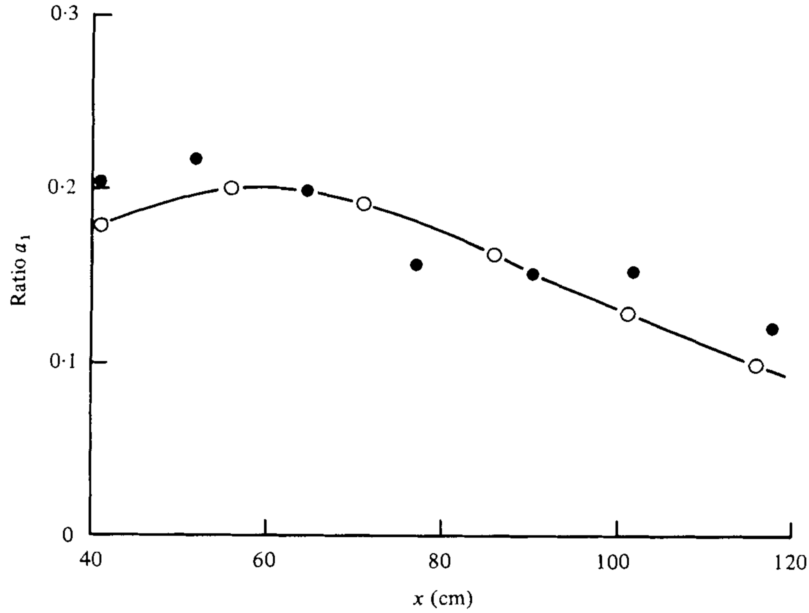


FIGURE 2. Measured and calculated ratios of Reynolds shear stress to total intensity at the positions of maximum shear stress in the distorted wake flow. ○, measured values; ●, calculated values.

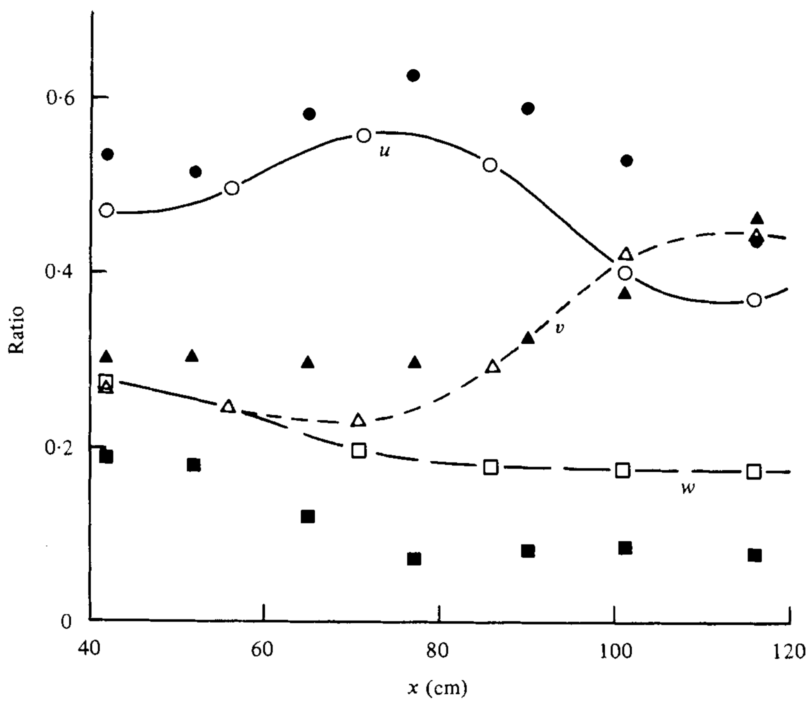


FIGURE 3. Measured and calculated ratios of component intensities to total intensity at positions of maximum shear stress in the distorted wake flow. ●, ○, $\overline{u^2}/\overline{q^2}$; ▲, △, $\overline{v^2}/\overline{q^2}$; ■, □, $\overline{w^2}/\overline{q^2}$ the open symbols denote measured values.

(*a*) *Irrotational strain ratios*

x (cm)	Ox strain	Oy strain	Oz strain
35	1	1	1
41	1·022	0·978	1
52	1·044	0·958	1
65	0·896	0·757	1·475
77	0·884	0·767	1·475
90	1·119	0·670	1·334
102	1·094	0·685	1·334
109	0·984	1	1·016
118	0·983	1	1·017

(*b*) *Shear steps*

x (cm)	Sequence A	Sequence B
35	1·5–0·5	2–0·5
41	1·5–0·4–0·3	1·9–0·4–0·3
52	1·5–0·3–0·2–0·2	1·8–0·3–0·2–0·2
65	1·5–0·2–0·2–0·2–0·1	1·8–0·2–0·2–0·2–0·1
77	1·5–0·2–0·2–0·2–0·1–0·1	1·7–0·2–0·2–0·2–0·1–0·1
90	1·5–0·2–0·2–0·1–0·1–0·1–0·1	1·7–0·2–0·2–0·1–0·1–0·1–0·1
102	1·5–0·3–0·2–0·1–0·1–0·1–0·1–0·1	1·5–0·3–0·2–0·1–0·1–0·1–0·1–0·1
109	1·5–0·2–0·2–0·2–0·1–0·1–0·1–0·1–0·1	1·4–0·2–0·2–0·2–0·1–0·1–0·1–0·1–0·1
118	1·5–0·2–0·2–0·2–0·1–0·1–0·1–0·1–0·1–0·1	1·3–0·2–0·2–0·2–0·1–0·1–0·1–0·1–0·1–0·1

Note. For any value of x, a distortion sequence is an initial simple shear equal to the first of the shear steps for that position, followed by the irrotational strain for $x = 35$, followed by the next shear step and repeating until the shear steps give out.

Table 1. Distortion sequences

In figures 2 and 3, the ratios of the Reynolds stresses to the total intensity near the positions of maximum shear stress are compared with values calculated from the rapid distortion approximation. Two kinds of distortion history were used. One supposes that the turbulence before entering the distorting section has been sheared by a fixed amount, and that passage through the duct imposes further plane shear in addition to the irrotational distortion of the ambient flow. The other makes an allowance for entrainment or finite eddy lifetime by keeping constant total shear at all stages. The calculation is necessarily done in steps, alternately a plane shear followed by an irrotational distortion, and two typical step sequences are listed in table 1.

The calculations showed that the ratios, particularly the shear stress ratios, were not sensitive to changes in the history of shearing, to the number of the steps or to the eddy transfer coefficient, and the values in figure 2 are averages for several step sequences and values of the transfer coefficient. At entry to the distorting section, the rate of shearing was about $150\,s^{-1}$, decreasing to about $20\,s^{-1}$ at exit, compared with transverse rates of strain of approximately $25\,s^{-1}$ within the distortion. Passage through the distortion took nearly 0·06 s.

The calculated values of the shear stress ratio are mostly higher than the measured values, a discrepancy that can be attributed to smaller quasi-isotropic eddies, and the variations of the ratio are well described. Of particular interest is the initial rise in the ratio as the flow accelerates before entering the distorting section. The rise, predicted by the rapid distortion model and confirmed by these measurements, shows that longitudinal extension of a sheared flow produces changes of turbulence structure which may exert a strong effect on a flow.

Ratios of the normal Reynolds stresses to the total intensity are less well described, but again their general variations are reproduced (figure 3).

4. Flow in a curved mixing layer

Recent measurements of turbulence in a curved mixing layer (Castro & Bradshaw 1976) have shown that the shear stress ratio decreases to nearly half its 'normal' value after a change of direction by 45° but recovers to nearly its original value just before becoming a plane layer moving at 90° to its original direction. The sense of the curvature is to make the mean flow stable to fluid displacements with conservation of angular momentum, analogous to the effect of a stabilizing density gradient in horizontally stratified flow, and the initial decrease in the ratio is expected. The surprising feature is the recovery towards the value for a plane layer before flow curvature becomes negligible, but Elliott (1976) has shown that the rapid distortion model can give a qualitative description of the recovery. More recently, Savill (1979) has been able to obtain very good agreement between observed and calculated values of the ratio by allowing for the effects of entrainment during the flow development (figure 4).

The curious behaviour of the stress ratio may be explained as a consequence of the inertial waves that can propagate in rotationally stable flows. For simplicity, consider the effect of solid-body rotation on turbulence with an initial Reynolds shear stress. The rapid-distortion result (see appendix) is that, in a fluid rotating with angular velocity Ω about the Ox_2 axis, a_1 and a_3, the Ox_1 and Ox_3 components of the Fourier coefficient of wavenumber (k_1, k_2, k_3), are given by

$$\left.\begin{aligned} a_1 &= a_1'\left(\cos\lambda t - \frac{k_1k_3}{kk_2}\sin\lambda t\right) + a_3'\,\frac{k}{k_2}\left(1-\frac{k_1^2}{k^2}\right)\sin\lambda t, \\ a_3 &= -a_1'\frac{k}{k_2}\left(1-\frac{k_3^2}{k^2}\right)\sin\lambda t + a_3'\left(\cos\lambda t + \frac{k_1k_3}{kk_2}\sin\lambda t\right), \end{aligned}\right\} \tag{4.1}$$

where $\lambda = 2\Omega k_2/k$. Both a_1 and a_3 vary sinusoidally with time at radian frequency λ, and their contribution to the velocity product u_1u_3 oscillates with frequency 2λ about a mean value of

$$\tfrac{1}{2}(a_1'a_3'^* + a_1'^*a_3')\frac{k_1^2+k_3^2}{k_2^2} + \frac{k_1k_3}{k_2^2}\left(1-\frac{k_3^2}{k^2}\right)\,a_1'a_1'^* + \frac{k_1k_3}{k_2^2}\left(1-\frac{k_1^2}{k^2}\right)a_3'a_3'^*.$$

If the wavenumber is parallel to the axis of rotation, the product contribution oscillates at frequency 4Ω about a mean value of zero, but for other directions the frequencies are less and the mean values are not necessarily zero. If an initial Reynolds stress implies a degree of phase coherence between the contributions of different components, the coherence will disappear with time and the stress will eventually approach a limiting value. However, since the possible frequencies have a sharp upper

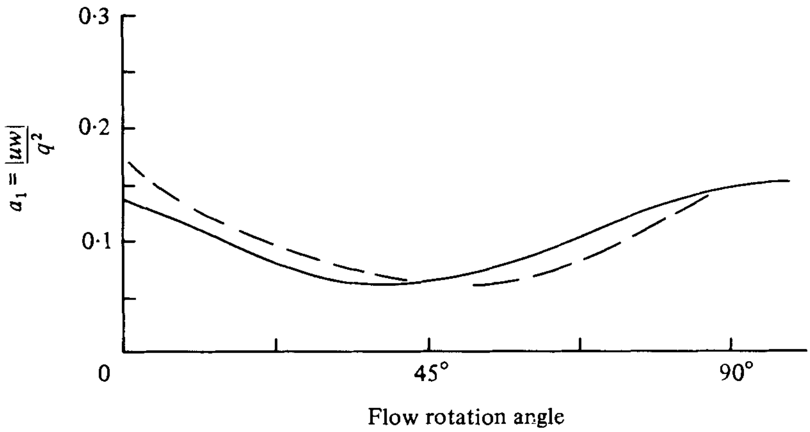


FIGURE 4. Comparison of measured and calculated ratios of Reynolds shear stress to total intensity for the curved mixing-layer. ——, observed by Castro & Bradshaw (1976); — — —, calculated by Savill (1979).

limit at 4Ω, the approach to the limit is likely to take the form of a damped oscillation with that frequency. In terms of flow direction, the oscillation of Reynolds stress goes through a *complete* cycle while the flow turns through 90°, and *recovery* occurs after turning through 45°.

In the curved mixing layer of Castro & Bradshaw, the rate of rotation of flow direction is considerably less than the rate of shearing and only some of the Fourier components can have amplitudes varying periodically in time, the remainder being aperiodic. The periodic components will cause an oscillation of Reynolds stress, and numerical calculations indicate that the frequency is near 2Ω for the particular ratio of rotation to shear. It seems likely that the recovery angle would increase as the ratio of shear to rotation rates increases.

5. Turbulent flow over water waves

From the initial work of Miles (1957), most attempts to calculate the rate of energy transfer from the mean flow of a deep turbulent boundary layer to water waves have led to values much smaller than those measured either in the laboratory or at sea. No noticeable improvement is obtained by including the effects of Reynolds stresses, whether their relation to the mean flow is described by a coefficient of eddy viscosity or by the equation for the turbulent energy assuming constant ratios of Reynolds stresses to turbulent energy. Recently, better agreement has been obtained by postulating a visco-elastic response of the stresses to the wave-induced distortion, or by assuming large variations of roughness over the wave surface (Gent & Taylor 1976). However, a considerable increase in the energy transfer is predicted if the stress ratios change in the way described by the rapid-distortion equations of motion without having to introduce parameters of critical but uncertain magnitude.

Except for the use of variable stress-ratios, the calculation of the pressure field over the waves follows closely that outlined in an earlier paper (Townsend 1972), the other change being inclusion in the energy equation of a term describing work done by the mean flow variations on normal Reynolds stresses of the basic flow. Briefly, the motion is described in a co-ordinate system moving with the surface wave, and the mean velocity has components $(U_0+u, 0, w)$ for a wavenumber in the Ox direction (Oz is the vertical direction). The wave-induced motion is treated as a small perturbation of the constant-stress turbulent flow over a plane surface with roughness length z_0, so that the unperturbed velocity distribution is

$$U_0(z) = \frac{\tau_0^{\frac{1}{2}}}{K}\log z/z_0 - c, \tag{5.1}$$

where τ_0 is the (kinematic) constant shear stress, and c is the phase velocity of the wave. Then the equations of continuity and momentum are, to the approximation of small disturbance,

$$\left.\begin{aligned} \partial u/\partial x + \partial w/\partial z &= 0, \\ U_0\,\partial u/\partial x + w dU_0/dz &= -\partial P/\partial x + \partial\tau/\partial z + \partial\tau_n/x, \\ U_0\,\partial w/\partial x &= -\partial P/\partial z + \partial\tau/\partial x, \end{aligned}\right\} \tag{5.2}$$

where P is the difference of the mean pressure and the normal Reynolds stress τ_{33}, τ is the difference of the local shear stress τ_{13} from the undisturbed value τ_0, τ_n is the change of $(\tau_{11}-\tau_{33})$, the difference of the normal stresses from the unperturbed value.

To the approximation of small changes in flow quantities, the equation for the turbulent kinetic energy, $\frac{1}{2}\overline{q^2}$, is

$$U_0\,\partial/\partial x(\tfrac{1}{2}\overline{q^2}) + \partial/\partial z(\overline{pw}+\tfrac{1}{2}\overline{q^2w})' = (\tau_{11}-\tau_{33})\partial u/\partial x + \tau dU_0/dz + \tau_0\,\partial u/\partial z - \epsilon', \tag{5.3}$$

where ϵ' is the difference of the local rate of energy dissipation from its unperturbed value. To use the energy equation, it is necessary to provide relations between the turbulent energy and the Reynolds stresses, here through the rapid-distortion model, and for the rates of energy dissipation and vertical transport of energy in the perturbed flow. As in the earlier calculation, it is assumed that the dissipation length parameter is $L_\epsilon = Ka_1^{-\frac{3}{2}}\,(z-\zeta e^{-kz})$, where a_1 is the ratio of shear stress to turbulent intensity, $\tau_{13}/\overline{q^2}$, in the undisturbed flow, and ζ is the local elevation of the water surface. By definition, $L_\epsilon = (\overline{q^2})^{\frac{3}{2}}/\epsilon$ and, to the approximation used,

$$\epsilon' = \frac{\tau_0^{\frac{3}{2}}}{Kz}\left[\frac{3}{2}\frac{(\overline{q^2})'}{\overline{q^2}} + \frac{\zeta}{z}\,e^{-kz}\right].$$

The changes in the vertical flux of turbulent energy are conveniently described by a diffusion coefficient proportional to the local 'eddy viscosity', so that the change is

$$(\overline{pw}+\tfrac{1}{2}\,\overline{q^2w})' = -\beta K\tau_0^{\frac{1}{2}}z\frac{\partial}{\partial z}\,(\tfrac{1}{2}\overline{q^2}).$$

The value of the parameter β is not critical.

Far above the surface, the time scales of the undisturbed flow are long compared with the time scale of the wave perturbation following the mean flow, and the rapid-distortion approximations are valid. There, the variations of the stress-intensity ratios are given by

$$U_0\,\partial/\partial x(\tau_{13}/\overline{q^2}) = A_1\,\partial u/\partial z + A_2\,\partial w/\partial x + A_3\,\partial u/\partial x,$$

$$U_0\,\partial/\partial x(\tau_n/\overline{q^2}) = B_1\,\partial u/\partial z + B_2\,\partial w/\partial x + B_3\,\partial u/\partial x,$$

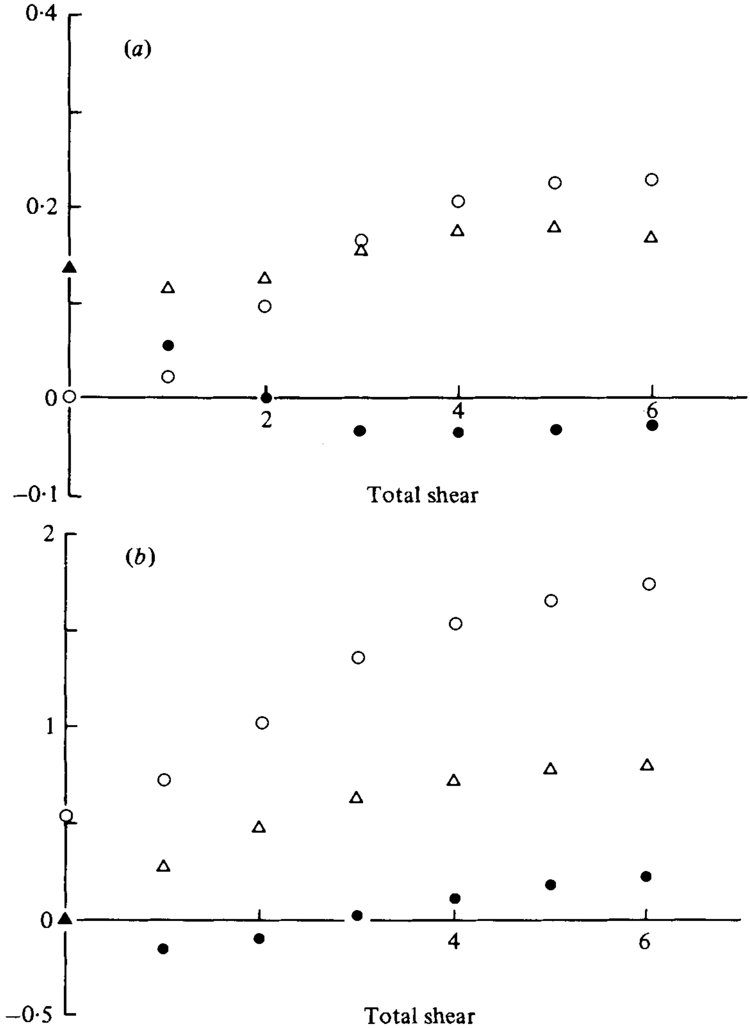


FIGURE 5. Incremental responses of stress-ratios to additional small strains after a finite plane shear: (a) response of shear-stress ratio, $a_1 = |\overline{uw}|/\overline{q^2}$; (b) response of normal-stress ratio, $(\overline{u^2}-\overline{w^2})/\overline{q^2}$. ●, response to additional shear in Ox direction; +, response to additional shear in Oz direction; ○, response to irrotational distortion with principal axes Ox and Oz.

where the coefficients A, B are the incremental rates of change for suddenly imposed additional distortions, A_1, B_1 for additional shearing in the Ox direction, A_2, B_2 for additional shearing in the Oz direction, and A_3, B_3 for additional irrotational distortion in the xOz plane with principal axes Ox and Oz. Near the surface, time scales of the basic flow are small and the stress-intensity ratios remain near their equilibrium values, a_1 and a_n, and are almost unaffected by the perturbation.

The simplest way to interpolate between the two extreme regions is to write the equations for the stress ratios as

$$\left.\begin{aligned} U_0\,\partial/\partial x(\tau_{13}/\overline{q^2})+\lambda(\tau_{13}/\overline{q^2}-a_1) &= A_1\,\partial u/\partial z+A_2\,\partial w/\partial x+A_3\,\partial u/\partial x,\\ U_0\,\partial/\partial x(\tau_n/\overline{q^2})+\lambda(\tau_n/\overline{q^2}-a_n) &= B_1\,\partial u/\partial z+B_2\,\partial w/\partial x+B_3\,\partial u/\partial x,\end{aligned}\right\} \tag{5.4}$$

where λ is a relaxation rate proportional to the rate of shear in the unperturbed flow. If $\lambda \gg kU_0$, the ratios have their equilibrium values, while, if $\lambda \ll kU_0$, their changes are given by the rapid-distortion calculations. λ can be considered as the reciprocal of the effective lifetime of eddies at the particular level and, for consistency, it should be set equal to $(1/\alpha_0)\,dU_0/dz$ where α_0 is the effective total shear of the undisturbed flow.

Using the rapid-distortion equations, values of the incremental rates have been calculated for total shears up to six and several values of the eddy transfer coefficient (figure 5). Possibly the most interesting feature is the large response of the shear-stress to intensity ratio, $\tau_{13}/\overline{q^2}$, to irrotational distortion with principal axes Ox and Oz, i.e. to flow acceleration. The effect is that acceleration causes an increase in the ratio, in agreement with measurements in a distorted wake (see § 3), and it is apparent that the elastic response of boundary layer turbulence to additional strain is highly anisotropic.

The set of linearized equations (5.2)–(5.4) have been solved numerically with the boundary conditions:

(i) that pressure, stress and velocity changes are small high above the wave surface;

(ii) that the flow velocity close to the surface is parallel to it and described by the logarithmic velocity distribution,

$$U - U_s = \frac{(\tau_0+\tau)^{\frac{1}{2}}}{K}\ln(z-\zeta)/z_0 \tag{5.5}$$

where U_s is the surface velocity, $(\tau_0+\tau)$ is the local shear stress at the surface, and $\zeta = \zeta_0 \exp ikx$ is the displacement of the surface from the horizontal plane, $z = 0$. On the grounds that rapid-distortion calculations for a total shear of six give values of a_n and a_1 near the observed values, the coefficient λ has been set to

$$\lambda = \tfrac{1}{6}dU_0/dz = \tfrac{1}{6}\tau_0^{\frac{1}{2}}/(Kz) \tag{5.6}$$

and the values of the A and B coefficients are those for $\alpha_0 = 6$ and an eddy transfer coefficient of $\beta = 0{\cdot}3$.

Figures 6 and 7 show the results of calculations for $\ln kz_0 = -8$, -10 and a range of values of $c/\tau_0^{\frac{1}{2}}$, expressed in non-dimensional form using scales formed from τ_0 and k. They present the values of the complex amplitudes of the pressure and stress variations, $(P_{\mathscr{R}} + iP_{\mathscr{I}})$ and $(\tau_{\mathscr{R}} + i\tau_{\mathscr{I}})$, such that

$$\left.\begin{aligned} P &= k\zeta_0\tau_0(P_{\mathscr{R}} + iP_{\mathscr{I}})\exp(ikx)\\ \text{and}\qquad \tau &= k\zeta_0\tau_0(\tau_{\mathscr{R}} + i\tau_{\mathscr{I}})\exp(ikx).\end{aligned}\right\} \tag{5.7}$$

The changes brought by the inclusion of changing stress-ratios are:

(i) the amplitudes of surface pressure variations are considerably increased;

(ii) the amplitudes of the surface stress variations are very much larger;

(iii) the region of flow modification extends much further above the wave surface;

(iv) the vertical distribution of shear-stress amplitude no longer bears a close resemblance to a diffusion wave.

It is not easy to isolate the mechanism that causes these considerable changes when the stress ratios vary. With the chosen flow parameters, the relaxation rates for turbulent energy and for the stress ratios are similar in magnitude, and elastic response

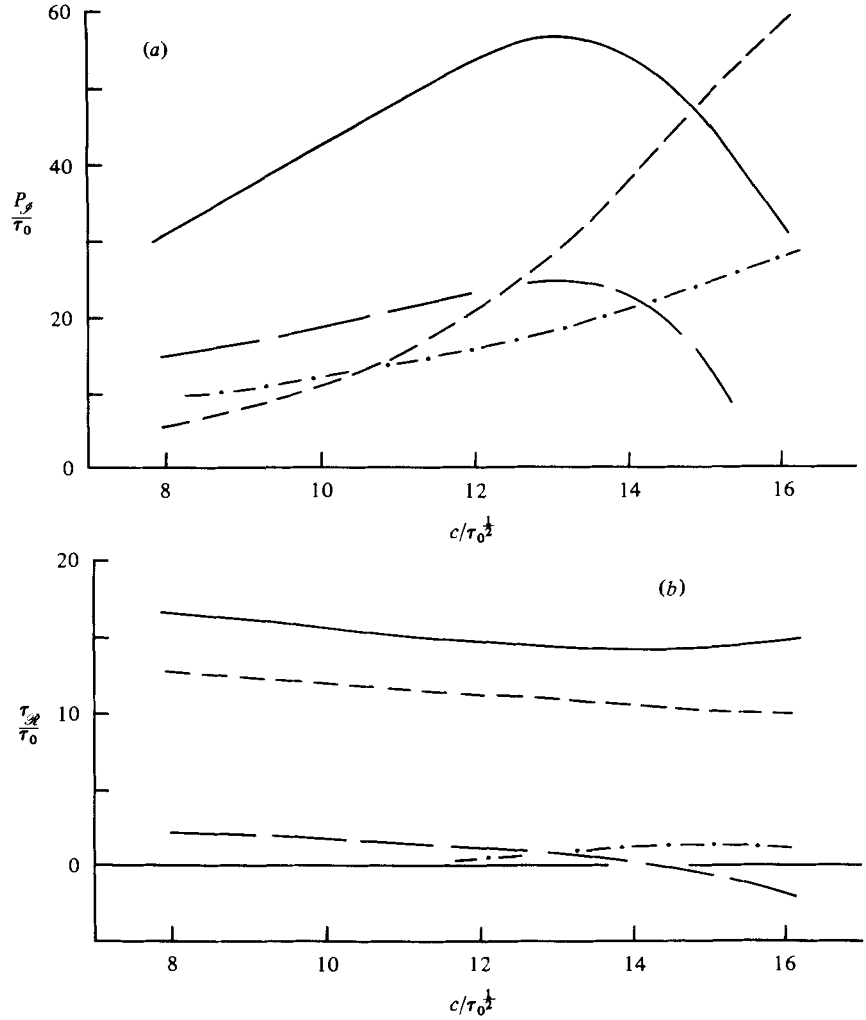
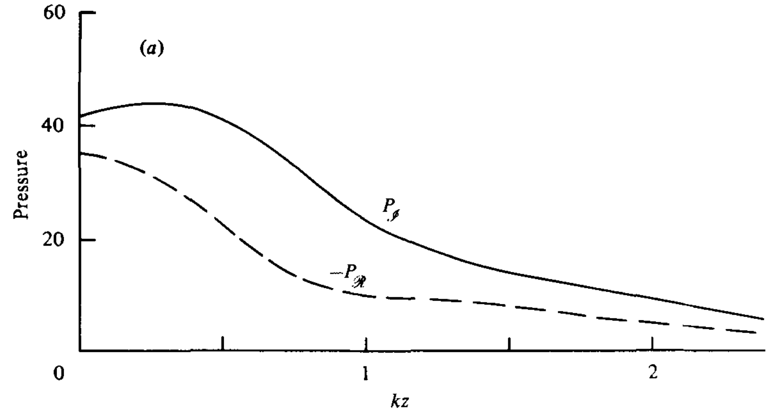
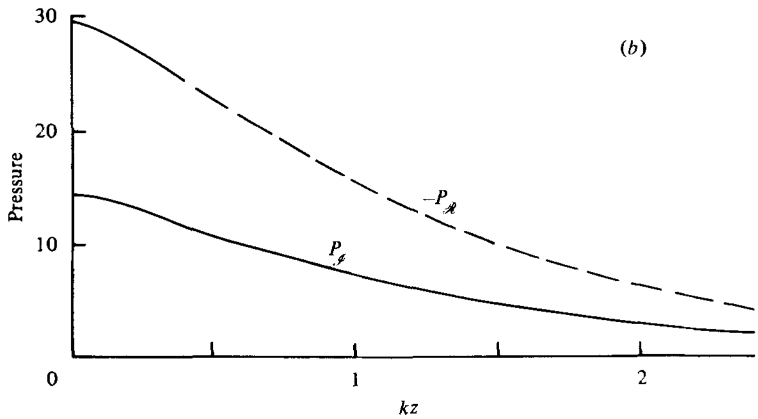


FIGURE 6. Amplitudes of surface pressure and shear-stress variations for a small-amplitude gravity wave in a deep turbulent boundary layer: (*a*) components of pressure amplitude in quadrature with wave displacement; (*b*) components of stress amplitude in phase with wave displacement. With variations of stress-ratios calculated for a total shear strain of six: ——, $KU(k^{-1})/\tau_0^{\frac{1}{2}} = 8$; ---, $KU(k^{-1})/\tau_0^{\frac{1}{2}} = 10$. With constant stress-ratios: — —, $KU(k^{-1})/\tau_0^{\frac{1}{2}} = 8$; —·—· $KU(k^{-1})/\tau_0^{\frac{1}{2}} = 10$.

will be dominant above the height, $z = 0{\cdot}5k^{-1}$. With constant stress-ratios, the elastic response is nearly confined to additional shear in the Ox direction and is described by shear modulus of $2a_1\tau_0$. With the variable stress-ratios, the response is highly anisotropic and components of the modulus tensor may be as large as $12\tau_0$. Possibly the increased pressure amplitudes and the greater depth of modified flow arise from the increased effective rigidity of the turbulent fluid.

Transfer of energy from the boundary layer to the water waves takes place through working of the surface pressures and through working of the surface shear stresses. Of these, the energy flux from the normal pressures is of magnitude $\rho(k\zeta_0)^2\tau_0 cP_{\mathscr{I}}$, and it is communicated directly to the wave motion. The flux from the surface shear



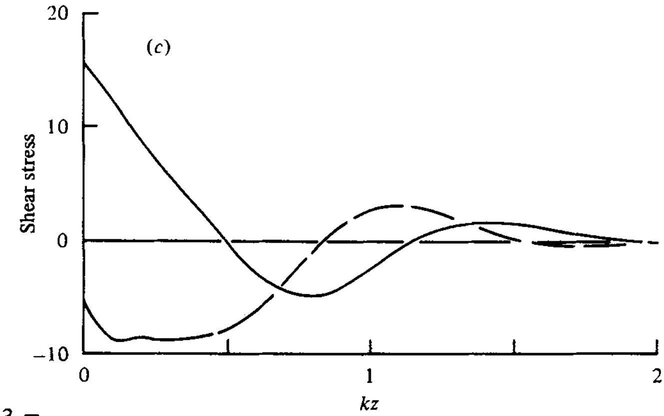


FIGURES 7(*a*) and (*b*). For legend see facing page.

stresses, $\rho(k\zeta_0)^2\tau_0 c\tau_{\mathscr{R}}$, is not transferred directly into wave energy but Longuet-Higgins (1969) has shown that a considerable part ends as wave energy. So the non-dimensional energy transfer for $\ln kz_0 = -8$ and $c/\tau_0^{\frac{1}{2}} = 10$ is in the range 42–58 if the effects of additional distortion on the stress ratios are included in the calculation, compared with 14–15 if they are ignored.

6. Discussion

Over the last twenty years, practical procedures for the calculation of flow development have become more realistic in that they now include moderately detailed descriptions of the turbulent motion, particularly as expressed in the Reynolds stress tensor. Less attention has been paid to the spatial structure of the turbulent motion in spite of the numerous experimental studies of organized eddies, possibly because of the difficulty of devising a compact, numerical specification. Current methods based on the Reynolds transport equations have been moderately successful by concentrating on the generation, transport, redistribution and decay of the Reynolds



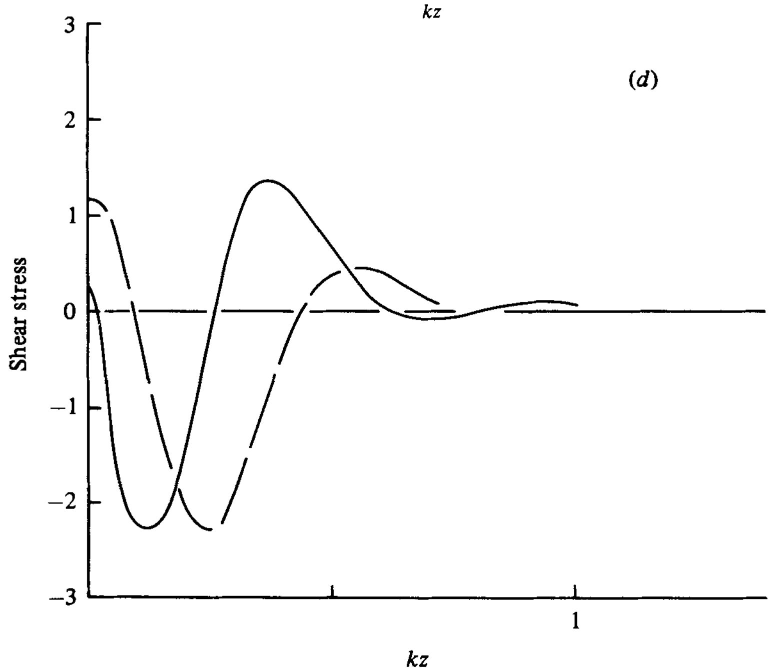


FIGURE 7. Variation with height of the complex amplitudes of pressure and shear-stress over waves for $c/\tau_0^{\frac{1}{2}} = 8$ and $KU(k^{-1})/\tau_0^{\frac{1}{2}} = 10$: (*a*) pressure amplitudes with variations of stress ratios; (*b*)pressure amplitudes with constant stress ratios; (*c*) shear-stress amplitudes with variations of stress ratios; (*d*) shear-stress amplitudes with constant stress ratios.

stresses, but the neglect of structural changes caused by mean flow distortion could be a serious defect for some applications. The rapid-distortion model emphasizes structure at the expense of firm predictions of turbulent intensity and, within that limitation, it may have a wider range of validity than the transport equation models.

The background to the rapid-distortion approach is the concept of an 'eddy' as a stable flow pattern, of limited spatial extent and discrete in the sense that one eddy does not overlap another to a significant extent. Then, nonlinear interactions between different eddies are weak and, within a single eddy, their effect is to stabilize it against disintegration into smaller eddies. Changes in eddy structure arise from interaction with mean flow gradients and are relatively unaffected by energy transfer to smaller eddies.

The rapid-distortion equations can be used to give an acceptably close description of the observed correlation functions in several nearly unidirectional shear flows, and, if this means that the calculated flow structures are similar to the real ones, the equations should be able to predict structure changes for small additional distortions of any kind. If the underlying assumptions are valid, the predictions should be useful for large distortions as well. Here, the comparison has been between measured and calculated values of the stress-intensity ratios, which are of the greatest practical interest. A more severe test would be to compare correlations, and measurements are now in progress.

The good agreement between measured and calculated stress ratios in the distorted wake and the curved mixing layer shows that the rapid-distortion model can describe the changes in the shear-stress ratio in flows subjected to additional distortion that is not simple shearing. Thus, longitudinal extension of a previously sheared flow leads to a considerable increase in the shear–stress ratio as well as the expected changes in ratios of the normal stresses, and solid-body rotation causes a decrease in the ratio followed by recovery towards the original value. Both the effects seem to depend on the initial structure of the turbulence and are not easily associated with source terms in the transport equation for the Reynolds stress.

Comparison between the rapid-distortion model and calculation schemes using transport equations is not a simple matter since they approach the problem from different angles. The rapid-distortion model sets out only to predict stress ratios and, if it were to form part of a calculation scheme, another equation would be needed to determine intensities. One possibility is to use an equation for the turbulent kinetic energy as in the calculation of flow over water waves in § 5. There are two reasons why it might be advantageous to incorporate the rapid-distortion model into a calculation scheme by adding the transport equation for the kinetic energy. One is that, in flows undergoing three-dimensional distortion, calculation might be simplified by having only one transport equation. The more important is that the rapid-distortion model offers a more complete description of the turbulence and can predict changes in stress ratios arising from unusual flow distortions.

The work described was done in the Cavendish Laboratory, assisted by grants from the Science Research Council and the Central Electricity Generating Board.

Appendix. Fourier coefficient and wavenumber transformations calculated from the rapid-distortion approximation for finite, simple distortions and rotations

(a) Simple shearing in the xOz plane

For a mean flow $(U, V, W) = (\alpha z, 0, 0)$, the Fourier coefficients of the velocity field change as

$$a_i(\mathbf{k}) = A_{ij}\, a_j'(\mathbf{k}'),$$

where dashes denote values before the distortion, and

$$A_{ij} = \begin{pmatrix} 1 & 0 & \dfrac{k_0'^2}{k_1(k_1^2+k_2^2)^{\frac{3}{2}}}\,(k_2^2 P - k_1^2 Q) \\ 0 & 1 & \dfrac{-k'^2 k_2}{(k_1^2+k_2^2)^{\frac{3}{2}}}\,(P+Q) \\ 0 & 0 & k'^2/k^2 \end{pmatrix},$$

where†
$$P = -\arctan\left(\frac{\alpha k_1(k_1^2+k_2^2)^{\frac{1}{2}}}{k^2-\alpha k_1 k_3'}\right),$$
$$Q = \frac{-\alpha k_1(k_1^2+k_2^2)^{\frac{1}{2}}\,(k'^2-2k_1^2+\alpha k_1 k_3')}{k'^2k^2}$$

and
$$k^2 = k_1^2+k_2^2+k_3^2.$$

The changes in wavenumber for a component are given by
$$k_1 = k_1', \quad k_2' = k_2', \quad k_3 = k_3'-\alpha k_1',$$

and, if the rate of shear is uniform in time, the wavenumber integral for the transfer term is
$$\int_0^{t_d} k^2 dt = \frac{\alpha}{\dot{\alpha}}\,(k'^2-\alpha k_1' k_3'+\tfrac{1}{3}\alpha^2 k_1'^2).$$

Transformations for other forms of simple shearing are easily found by appropriate change of the suffices.

(*b*) *Irrotational distortion with fixed principal axes*

If α, β and γ are the extension ratios in the principal, axial directions, Ox, Oy, Oz (where $\alpha\beta\gamma = 1$), the changes in wavenumber of a Fourier component are
$$k_1 = \alpha^{-1}k_1', \quad k_2 = \beta^{-1}k_2', \quad k_3 = \gamma^{-1}k_3',$$

and the transformation matrix for the components is
$$A_{ij} = k^{-2}\begin{pmatrix} k'^2+c_{32}k_2'^2+c_{23}k_3'^2 & -c_{32}k_1'k_2' & -c_{23}k_1'k_3' \\ -c_{31}k_1'k_2' & k'^2+c_{31}k_1'^2+c_{13}k_3'^2 & -c_{13}k_2'k_3' \\ -c_{21}k_1'k_3' & -c_{12}k_2'k_3' & k'^2+c_{21}k_1'^2+c_{12}k_2'^2 \end{pmatrix},$$

where $k^2 = k_1^2+k_2^2+k_3^2$, and $C_{12} = (1-\alpha/\beta)$, $C_{23} = (1-\beta/\gamma)$, etc.

For a uniform rate of straining, $\dot{\alpha}/\alpha$ = constant, $\dot{\beta}/\beta$ = constant, $\dot{\gamma}/\gamma$ = constant,
$$\int_0^{t_d} k^2 dt = \frac{1}{2}\frac{\alpha}{\dot{\alpha}}\,(1-\alpha^{-2})\,k_1'^2+\frac{1}{2}\frac{\beta}{\dot{\beta}}(1-\beta^{-2})\,k_2'^2+\frac{1}{2}\frac{\gamma}{\dot{\gamma}}(1-\gamma^{-2})\,k_3'^2.$$

(*c*) *Irrotational distortion with axisymmetric, constant circulation flow*

For this distortion, the mean flow is (locally) constant along circular streamlines with velocity along the streamlines,
$$U = K/r,$$

where r is distance from the centre of curvature.

With respect to axes with Ox in the local direction of mean flow, Oz radially inwards, and Oy parallel to the axis of symmetry, the change in wavenumber is described by
$$k_1' = k_1', \quad k_2 = k_2', \quad k_3 = k_3'-2\theta k_1',$$

† The arctan function should be interpreted as the direction in the range π to $-\pi$, of a two-dimensional vector whose components are the denominator and numerator of the argument.

where θ is the total change in direction of the flow. The transformation matrix for the Fourier components is

$$A_{ij} = k^{-2}\begin{pmatrix} k'^2 - 2\theta k_1' k_3' & 0 & 2\theta k_1'^2 \\ 2\theta k_2' k_3' & k^2 & 0 \\ -2\theta k_2'^2 & 2\theta k_1' k_2' & k'^2 \end{pmatrix}.$$

For a uniform rate of rotation

$$\int_0^{td} k^2 dt = \frac{\theta}{\dot{\theta}}(k'^2 - 2\theta k_1' k_3' + \theta^2 k_1'^2).$$

(d) *Solid body rotation*

If the mean flow is a solid-body rotation, wavenumbers of components do not change in the rotating system of co-ordinates, and individual components vary sinusoidally in time with radian frequency, $\lambda = 2\Omega k_2/k$, where Ω is the angular velocity of rotation about the Ox_2 axis (positive if anti-clockwise in the $x_1 Ox_3$ plane). To satisfy the initial conditions, the transformation matrix is

$$A_{ij} = \cos\lambda t\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin\lambda t\begin{pmatrix} -k_1 k_3/(kk_2) & 0 & (1-k_1^2/k^2)\,k/k_2 \\ k_3/k & 0 & -k_1/k \\ -(1-k_3^2/k^2)\,k/k_2 & 0 & k_1 k_3/(kk_2) \end{pmatrix}.$$

Since the wavenumber of a component is not changed,

$$k^2 dt = k^2\theta/\dot{\theta}$$

where $\theta = \Omega t$ is the angle turned.

(e) *Calculation for complex distortions*

To show that the rapid distortion equations lead to different transformations of the velocity field if the same, *rotational* distortion is accomplished by different distortion paths, it is sufficient to consider the effect of a solid-body rotation through a right-angle combined with a finite simple shear of magnitude α. The total distortion does not depend on the order, but the change in the Fourier coefficients of the velocity field does. For the rotation, the transformation matrix is

$$R_{ij} = \begin{pmatrix} -k_1 k_3/(kk_2) & 0 & (1-k_1^2/k^2)\,k/k_2 \\ k_3/k & 0 & -k_1/k \\ -(1-k_3^2/k^2)\,k/k_2 & 0 & k_1 k_3/(kk_2) \end{pmatrix}$$

with no change in wavenumber. For the shear, the matrix is

$$S_{ij} = \begin{pmatrix} 1 & 0 & X \\ 0 & 1 & Y \\ 0 & 0 & k^2/k_n^2 \end{pmatrix},$$

where $k_n^2 = k^2 - 2\alpha k_1 k_3 + \alpha^2 k_1^2$, and X, Y are complex expressions (see § 1).

For rotation followed by shear, the first element of the total transformation matrix, $S_{ip}R_{pj}$, is

$$A_{11} = -k_1(k_3 - \alpha k_1)/(k_n k_2) - (k_1^2 + k_2^2)X/(k_n k_2),$$

where the wavenumber components have their original values. For shear followed by rotation, the matrix is $R_{ip}S_{pj}$ and the first element is

$$A_{11} = -k_1(k_3 - \alpha k_1)/(k_n k_2),$$

not at all the same.

To calculate the velocity transformation for a continuously complex distortion, the equations for A_{ij} could be integrated numerically, but adequate results can be obtained by using a sequence of moderately small distortions of the simple types. Both for economy of effort and for accuracy, the types should be chosen so that the step-wise total distortion should always be close to the continuous total distortion and so that no step has the effect of cancelling part of the effect produced by the previous step. For example, solid-body rotation should be used rather than constant-circulation rotation for the curved mixing layer since the actual mean velocity field is inertially more stable than a solid-body rotation. On the other hand, for a wake developing in a strongly curved, irrotational flow, the use of solid-body rotation steps would be wasteful and inaccurate, particularly on the unstable half of the flow.

If the mean flow is three-dimensional, more than three types of distortion may be necessary but the dominant distortion in many flows is commonly simple shearing in the flow direction combined with smaller amounts of irrotational distortion and a transverse simple shear. A possible step-sequence might be: longitudinal simple shear, transverse shear, irrotational change of direction, fixed-axes irrotational distortion.

REFERENCES

Batchelor, G. K. & Proudman, I. 1954 *Quart. J. Mech. Appl. Math.* **7**, 83.

Bradshaw, P. 1969 *J. Fluid Mech.* **36**, 177.

Castro, I. P. & Bradshaw, P. 1976 *J. Fluid Mech.* **73**, 265.

Deissler, R. G. 1965 *Phys. Fluids* **8**, 391.

Elliott, C. J. 1976 Ph.D. dissertation, University of Cambridge.

Gent, P. R. & Taylor, P. A. 1976 *J. Fluid Mech.* **77**, 105.

Hunt, J. C. R. 1973 *J. Fluid Mech.* **61**, 625.

Launder, B. E., Reece, G. J. & Rodi, W. 1975 *J. Fluid Mech.* **67**, 569.

Longuet-Higgins, M. S. 1969 *Phys. Fluids* **12**, 737.

Miles, J. W. 1957 *J. Fluid Mech.* **3**, 185.

Moffatt, H. K. 1965 The interaction of turbulence with strong wind shear. In *Proc. URSI-IUGG Coll. on Atmos. Turbulence & Radio Wave Propag.* Moscow: Nauka.

Pearson, J. R. A. 1959 *J. Fluid Mech.* **5**, 274.

Savill, A. M. 1979 Ph.D. dissertation, University of Cambridge.

Townsend, A. A. 1970 *J. Fluid Mech.* **41**, 13.

Townsend, A. A. 1972 *J. Fluid Mech.* **55**, 719.

Townsend, A. A. 1976 *The Structure of Turbulent Shear Flow.* Cambridge University Press.

Tucker, H. J. & Reynolds, A. J. 1968 *J. Fluid Mech.* **32**, 657.